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Universal bifurcation property of two- or higher-dimensional dissipative systems in parameter space: why does 1D symbolic dynamics work so well?

H P Fang†

CCAST (World Laboratory) PO Box 8730, Beijing, 100080, People's Republic of China
Institute of Theoretical Physics, PO Box 2735, Beijing, 100080, People's Republic of China

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Abstract. The universal bifurcation property of the Hénon map in parameter space is studied with symbolic dynamics. The universal-L region is defined to characterize the bifurcation universality. It is found that the universal-L region for relatively small L is not restricted to very small b values. These results show that the fact that universal sequences with short period can be found in many nonlinear dissipative systems is also a universal phenomenon.

1. Introduction

One of the standard ways of investigating the dynamics of physical systems is by exploiting their universal (system-independent) property [1–8]. The best understood transition sequence is the period-doubling cascade, which has been observed in a variety of physical systems. Beyond the accumulation point for the period-doubling sequence there is chaos. Two decades ago Metropolis *et al* [1] that there is an ordered sequence of distinct periodic windows, each of which occurs for some range of control parameter, within the chaotic region for unimodal maps, $x_{n+1} = f(\mu, x_n)$. They have called this sequence the U-sequence since the ordering of the windows is system independent. Remarkably, this universality is also observed in systems with many degrees of freedom both experimentally [2, 3, 8] and theoretically [4–7] although the phase portraits of these two- or high-dimensional system still exhibit very complex behaviour which is clearly not one-dimensional or close to one-dimensional. It has been found that the periodic windows interspersed in chaotic region for these systems are ordered in a systematic way that is similar to those of one-dimensional (1D) maps. The most striking and detailed observation is obtained in the Lorenz equations:

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= rx - xz - y \\ \dot{z} &= xy - 8z/3. \end{aligned} \tag{1}$$

On the parameter r axis with $45 < r < 400$, all the 68 periodic windows of the Lorenz equations found can fit into those of a 1D antisymmetrical map with only one exception [9]. Experimentally, even though the Belousov–Zhabotinskii reaction involves more than 30 chemical species, it exhibits complex bifurcation behaviour that is modelled well by 1D maps [3].

† Mailing address: Department of Physics, Fudan University, Shanghai, 200433, People's Republic of China.

Despite these numerical and experimental observations, the underlying mechanism for the universal property is not fully understood. The motivation of this paper is to present an approach towards interpreting all these experimental and numerical observations and exploring their limitations. We will take the Hénon map [10]

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + y_n \\ y_{n+1} = bx_n \end{cases} \tag{2}$$

as an example. The bifurcation structure of the Hénon map in the two-dimensional parameter (a, b) space has been extensively discussed [11, 12]. In this paper, we will use symbolic dynamics [1, 14–20] of 1D mappings and 2D mappings to illustrate the universal topological property of the Hénon map at selected parameters by considering the unstable periodic orbits embedded in its chaotic attractor. Two topological quantities δ and L are defined to characterize this universal topological property. Then we discuss the universal bifurcation property of the Hénon map in 2D parameter space (a, b) by defining universal-L regions in which the Hénon map exhibits 1D bifurcation behaviour to period L. It is remarkable to find that the universal-L region for relative small L is not restricted to very small b values. We will also present two examples of ordinary differential equations (ODEs), the Rössler equations [13] and the forced Brusselator [4], to demonstrate the validity and robustness of our approach. These results show that the fact that universal sequences with short period can be found in many experiments or numerical calculations on nonlinear dissipative systems is also a universal phenomenon.

The paper is organized as follows. In section 2, we review the basic property of 1D unimodal maps. The universal bifurcation property and its limitations of the Hénon map in the 2D parameter (a, b) space is studied in section 3. To demonstrate the validity of the method presented in section 3, the universal bifurcation property of the Rössler equations and the forced Brusselator in a definite parameter axis is investigated in section 4. Finally, in section 5 we give our conclusion.

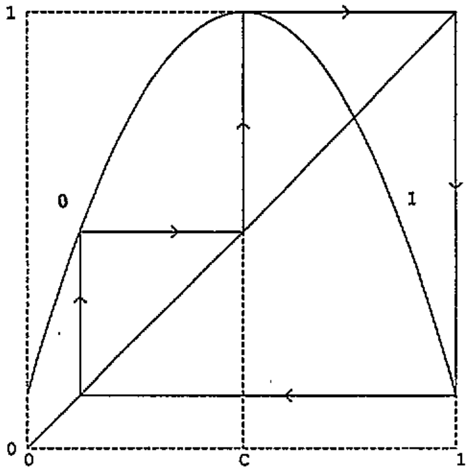

Figure 1. The logistic map $x_{n+1} = 1 - \mu x_n^2$ with $\mu == 1.754\,88$ exhibits a 3-cycle of the type 10C.

2. Universal sequences in 1D unimodal maps

By using the symbolic dynamics of 1D mappings, Metropolis *et al* (MSS) have already shown that the dynamics of unimodal 1D maps of the interval $[-1, 1]$ is embodied in the U-sequence of periodic windows [1, 14, 15]. Figure 1 shows a typical case. The extremum is denoted by a letter C. Each periodic window of the map can be labelled by a symbolic sequence of 0's and 1's that mark the location (to the left or right of C) of the successive iterates of the initial point C. For example, the only windows with period 5 are 101^2C, $10^2 1$C, and 10^3C (101^2C represents the periodic window $(101^2\mathrm{C})^\infty$ hereafter). Indeed, we can define an ordering [14, 15] for these symbolic sequences referring to the natural order in the interval $[-1, 1]$. These ordering rules are consistent with the ordering of a real number α defined for a sequence $\mathrm{S}(x)$ with an initial point x as follows [16]

$$\alpha(\mathrm{S}(x)) = \sum_{i=1}^{\infty} \mu_i 2^{-i} \tag{3}$$

with [16]

$$\mu_i = \begin{cases} 0 \\ 1 \end{cases} \quad \text{for} \quad \begin{cases} \sum_{j=1}^{i} & s_j = 0 \\ \sum_{j=1}^{i} & s_j = 1 \end{cases} \quad (\mathrm{mod}\ 2).$$

Since the symbolic sequence K=S(C), also called the kneading sequence, acquires a maximal α in this metric representation, a symbolic sequence $\mathrm{S}(x)$ corresponds to a real trajectory if and only if it satisfies

$$\alpha(\sigma^m(\mathrm{S}(x))) \leqslant \alpha(\mathrm{K}) \qquad m = 0, 1, 2, \ldots \tag{4}$$

where σ denotes the shift operator. With this admissibility condition, we can generate all the admissible periodic orbits for a given kneading sequence K. The kneading sequence changes as the controlling parameter alters. Since kneading sequences correspond to orbits coming from C, they should also satisfy the above condition. Thus we obtain the admissibility condition for the K themselves: a symbolic sequence K can be a kneading sequence if and only if it satisfies

$$\alpha(\sigma^m(\mathrm{K})) \leqslant \alpha(\mathrm{K}) \qquad m = 0, 1, 2, \ldots. \tag{5}$$

When K is a periodic string, K corresponds to a periodic window. From equation (5), we can generate all the possible periodic windows. It can be checked that there are only three period-5 windows in those listed above.

With the ordering rules in equation (3), all periodic windows can be ordered to yield the U-sequence. In the logistic map, this U-sequence is consistent with the increasing μ order which is listed in table 1 up to period 7.

3. Universal sequences in 2D Hénon maps

The Hénon map (2) has been extensively studied by using symbolic dynamics [16–19]. The set of all 'primary' tangencies between stable and unstable manifolds determines a binary generating partition which divides the attractor into two parts marked by letters 0 and 1. Any trajectory is encoded by a bi-infinite string $\mathrm{S}(x) = \ldots s_{-m} \ldots s_{-1} s_0 \bullet s_1 s_2 \ldots s_n \ldots$, where s_n denotes a letter for the nth image, s_{-m} a letter for the mth preimage, each is either

Table 1. Symbolic sequences for periodic windows of the Hénon map along two different parameter axes and that for the forced Brusselator equations. The axis I (long dashes in figure 4) and axis II (dashes dots) are two axes in and out of the universal-7 region of the Hénon map, on which a complete and incomplete U-sequence is found respectively.

			a range			Brusselator[4]
No.	Period	Word	$b = 0$	Axis I	Axis II	ω
0	0	C	0–0.749	1.032 565–1.164 538	none	
	2	1C	0.75–1.249	1.164 539–1.332 034	none	0.45–0.544
	2 × 2	101C	1.25–1.367	1.332 035–1.409 865	none	0.545–0.5777
2	6	10111C	1.4747–1.47973	1.499 840–1.495 505	none	0.582 49–0.582 51
3	7	101111C	1.574 72–1.575 41	1.585 860–1.585 225	1.521 515–1.522 540	
4	5	1011C	1.6244–1.628 43	1.635 395–1.631 640	1.588 650–1.594 165	0.5845–0.5848
5	7	101101C	1.673 96–1.6744	1.679 020–1.678 610	1.651 665–1.652 290	
6	3	10C	1.75–1.768 53	1.763 210–1.743 840	1.776 860–1.792 900	0.5947–0.654
	2 × 3	10010C	1.768 54–1.777 22	1.772 255–1.763 215	1.792 920–1.800 525	0.6545–0.7025
7	7	100101C	1.8323–1.832 39	1.835 474–1.835 567	1.811 364–1.811 581	
8	5	1001C	1.860 59–1.861 36	1.863 083–1.863 841	1.843 810–1.844 705	0.7068–0.7115
9	7	100111C	1.8848–1.884 83	1.886 869–1.886 911	1.870 425–1.870 470	
01	6	10011C	1.907 26–1.907 36	1.909 005–1.909 120	1.894 790–1.894 915	0.718–0.7185
00	7	100110C	1.927 15–1.927 16	1.925 261–1.925 289	1.939 390–1.939 415	
12	4	100C	1.940 56–1.941 53	1.938 829–1.939 827	1.952 015–1.952 945	0.7345–0.792
13	7	100010C	1.953 71–1.953 71	1.952 122–1.952 138	1.964 545–1.964 555	
14	6	10001C	1.966 77–1.9668	1.968 064–1.968 099	1.956 760–1.956 795	none
15	7	100011C	1.977 179–1.977 184	1.978 379–1.978 385	1.967 795–1.967 795	
16	5	1000C	1.985 41–1.985 468	1.984 101–1.984 160	none	0.8259–0.8675
17	7	100001C	1.991 814–1.991 818	1.992 925–1.992 928	1.982 968–1.982 969	
18	6	10000C	1.996 375–1.996 379	1.995 150–1.995 153	none	0.9015–0.923
19	7	100000C	1.999 096	1.997 890–1.997 891	none	

0 or 1, the bold dot indicates the 'present' position. In order to extend the grammar for unimodal maps to this map, a 'backward' variable is defined as [16]

$$\beta(\mathrm{S}(x)) = \sum_{i=1}^{\infty} \nu_i 2^{-(i+1)} \tag{6}$$

with

$$\nu_i = \begin{cases} 0 \\ 1 \end{cases} \qquad \text{for} \begin{cases} \displaystyle\sum_{j=0}^{-i} (1 - s_j) = 0 \\ \displaystyle\sum_{j=0}^{-i} (1 - s_j) = 1 \end{cases} \qquad \text{(mod 2)} \qquad \text{for } b => 0$$

$$\nu_i = \begin{cases} 0 \\ 1 \end{cases} \qquad \text{for} \begin{cases} \displaystyle\sum_{j=0}^{-i} s_j = 0 \\ \displaystyle\sum_{j=0}^{-i} s_j = 1 \end{cases} \qquad \text{(mod 2)} \qquad \text{for } b < 0.$$

For this 2D map, each primary tangency C is associated with a bi-infinite kneading sequence K (with the first backward letter s_0 undetermined which may be 0 or 1) and two symmetrical points (α(K), β_-(K)) and (α(K), $\beta_+(\mathrm{K}) = 1 - \beta_-(\mathrm{K})$) in the symbolic plane corresponding to $s_0 = 0$ and 1 respectively [18]. Analogously to those in unimodal maps, for all admissible points (α, β) with $\beta \in [\beta_-(\mathrm{K}), \beta_+(\mathrm{K})]$, α should be less than $\alpha(\mathrm{K})$ and thus the pruning front [16] is obtained by cutting out rectangles $\{\alpha, \beta | \alpha > \alpha(\mathrm{K}), \beta \in [\beta_-(\mathrm{K}), \beta_+(\mathrm{K})]\}$ for all points on the partition. The union of these rectangles gives the fundamentally forbidden zone. Consequently, the grammar for a word admissible or forbidden in this map can be expressed as: a bi-infinite word is admissible if and only if all its shifts never fall into the fundamentally forbidden zone [16, 18]. It is clear that there are infinitely many kneading sequences (corresponding to infinitely many primary tangencies) in a 2D map to determine the admissibility condition for a word, while there is only one kneading sequence in a 1D map.

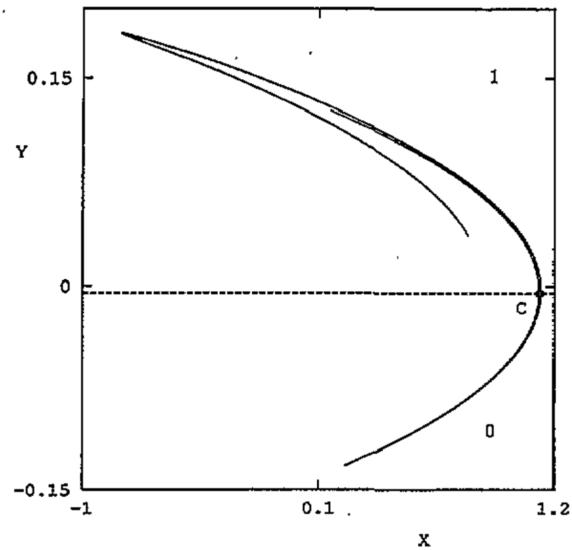

Figure 2. The strange attractor of the Hénon map for $(a, b) = (1.4, 0.16)$. The bold curves outline the strange attractor. The diamonds are the 'primary' tangencies. The dotted line connecting them divides the full attractor into two subsets marked by 0 and 1.

3.1. *Universality in the Hénon map*

Figure 3 shows a typical symbolic plane, $(a, b) = (1.4, 0.16)$. The corresponding attractor is shown in figure 2 which has a complicated structure. Its fractal dimension is 1.16 ± 0.03. Numerically 203 kneading sequences are found as shown in figure 2. It is found that the minimal and maximal of all the forward parts of these kneading sequences start with $\mathrm{K}_{\mathrm{min}} = 101111010101$ and $\mathrm{K}_{\mathrm{max}} = 101111011111$ respectively, corresponding to a minimal and maximal α-values $\alpha_{\mathrm{min}} = 0.837\,560$ and $\alpha_{\mathrm{max}} = 0.838\,466$ of all these kneading sequences. We define two quantities δ and L as

$$\delta = \alpha_{\mathrm{max}} - \alpha_{\mathrm{min}} = 0.000\,906 \tag{7}$$

$$L = -[\log_2 \delta] = 10 \tag{8}$$

where $[\log_2 \delta]$ denotes the integer part of $\log_2 \delta$. It is clear that $\delta = 0$ and $L \to +\infty$ in the 1D limit ($b = 0$). For $(a, b) = (1.4, 0.16)$, an unstable periodic orbit with length $n \leqslant L = 10$ cannot tell the difference between these kneading sequences. Indeed, no symbolic string with length $n \leqslant L$ lies in the interval between $\mathrm{K}_{\mathrm{min}}$ and $\mathrm{K}_{\mathrm{max}}$. Thus for

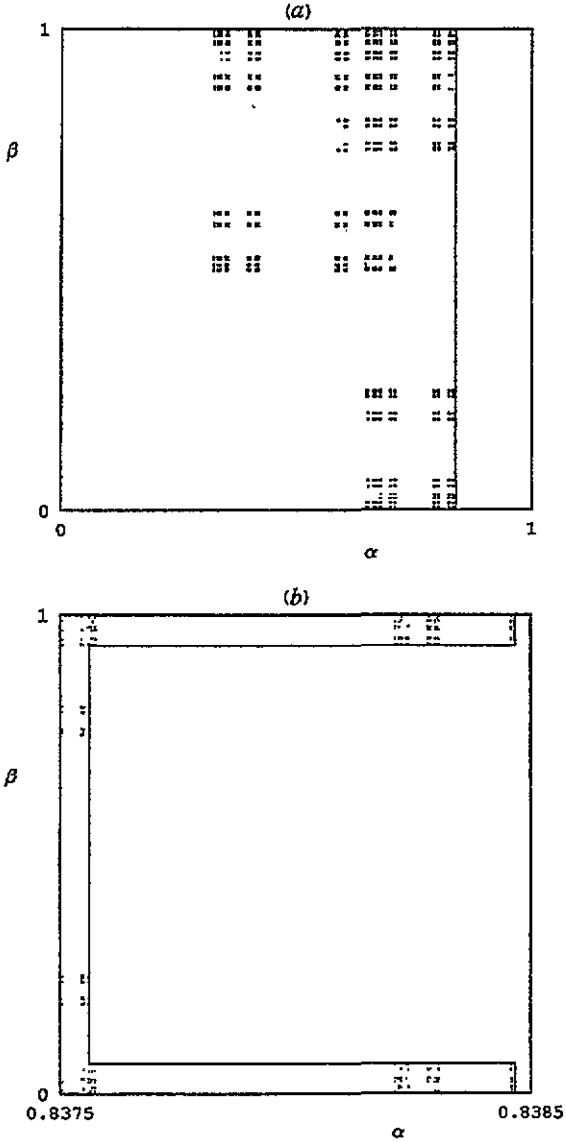

Figure 3. (*a*) The symbolic plane of the Hénon map for $(a, b) = (1.4, 0.16)$. (*b*) An enlarged part of the symbolic plane.

the unstable periodic orbits with length $n \leqslant L$, the grammar is completely determined by a symbolic string K_f, which is 1011110101 or 1011110111, the first 10 letters of K_{min} or K_{max}, that is, a word $S(x)$ corresponds to an unstable periodic orbit of the Hénon map for $(a, b) = (1.4, 0.16)$ if and only if it satisfies

$$\alpha(\sigma^m(S(x))) \leqslant \alpha(K_f) \qquad m = 0, 1, 2, \ldots. \tag{9}$$

This is just the grammar for unimodal maps with a kneading sequence K_f. Consequently the unstable periodic orbits of the Hénon map for $(a, b) = (1.4, 0.16)$ can be generated as that of unimodal maps with a kneading sequence K_f (see equation (4)). The only exception is the unstable periodic orbit K_f^{∞} which cannot be determined by equation (9). We note here that the Hénon map is divergent for $a = 1.4$ and $b > 0.315$.

As b decreases, L increases. It had already been shown [19] that $L = 32$ for $(a, b) = (1.4, 0.05)$. In the 2D phase space, even for $(a, b) = (1.4, 0.05)$ the attractor has a clear hook indicating that the map is two-dimensional. We emphasize that though the attractors reveal a very complicated structure in the 2D phase space, the topologies for these attractors may be very close to those in 1D maps in that the unstable periodic orbits can be generated with only one kneading sequence to some degree.

Now we consider the universal bifurcation property of the Hénon map in parameter space. Figures 4(a) and (b) show the isoperiodic lines [11, 12] for all the nine period-7 windows. Numerically we find that $L \leqslant 7$ for all the parameters a and b in the region between the two bold full curves shown in the figure†. We call this region the 'universal-7 region' hereafter. Thus all the periodic orbits with length $\leqslant 7$ of the Hénon map in this universal-7 region can be determined with only one kneading sequences as in those of 1D unimodal maps. Consequently, in this region there is a perfect MSS sequence up to period $\leqslant 7$ along any axis provided that the axis is never tangent to any isoperiodic lines. These axes are, in a sense, the same as the axis of $b = 0$ (corresponding to the logistic map). We present an example of these axes in figure 4 (line I, long dashes). The periodic windows on this axis are listed in table 1. It is clear that they share the same universal feature as 1D unimodal maps do up to period 7. We can also obtain the universal-M region numerically for $M = 5, 6, 8, 9, \ldots$ in which there are MSS sequences for period $\leqslant M$ along any axes provided that they are never tangent to any isoperiodic lines for period $\leqslant M$.

In figure 4 we also show that the borders for the Hénon map exhibit an attracting set with initial points (x_0, y_0) which are very close to the original point $(0, 0)$. Comparing these borders we can say that the universal-7 region is not restricted to very small b values. Thus it is very probable that we will obtain an MSS sequence for a relative short period (say, period 7) in the full 2D parameter plane of the Hénon map.

3.2. Incomplete U-sequence in the Hénon map

In fact, even on an axis outside the universal region, the Hénon map can exhibit approximately 1D behaviour if the axis is never tangent to any isoperiodic lines. In table 1 we also show the periodic windows on the axis represented by the dash-dotted line (II) in figure 4. It is clear that all of these words increase monotonically as a increases except the word 10001C and the period windows 10111C, 1000C, 10000C and 100000C are missing.

4. Applications to ODEs

The above idea can be extended to many other two- or higher-dimensional systems. Here we only take the Rössler's equations [13]

$$\begin{aligned} \dot{x} &= -y - x \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \tag{10}$$

† Though we can calculate the L value for given parameters, the distribution of L values are rather irregular so that we do not give iso-L lines. A detailed discussion will be presented elsewhere.

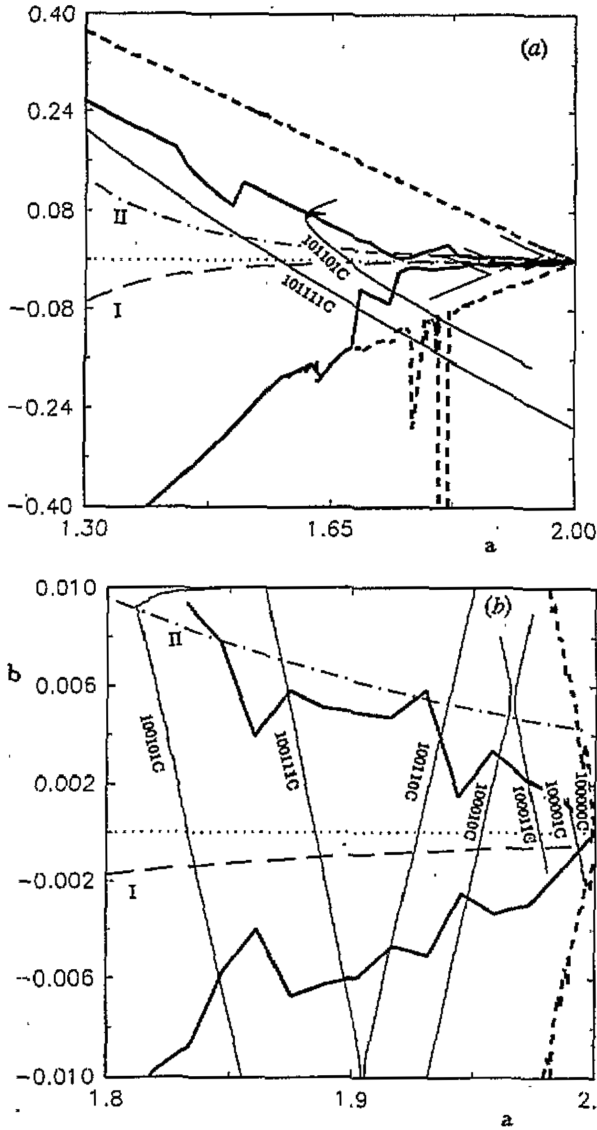

Figure 4. The isoperiodic lines together with the universal-7 region (the region between two bold curves) and the borders for the Hénon map exhibit an attracting set with initial points (x_0, y_0) very close to original point $(0, 0)$ (bold short dashes). The long dashes (I) and dashes-dot (II) represent two axes in and out of the universal-7 region, on which a complete and incomplete U-sequence is found respectively. (*b*) An enlarged part.

and the forced Brusselator [4]

$$\begin{aligned} \dot{x} &= A - (B+1)x - x^2 y + \alpha \cos(\omega t) \\ \dot{y} &= Bx - x^2 y \end{aligned} \qquad (11)$$

as examples.

The 2D attractor of the Rössler's equations is usually taken from a section of the 3D flow on the half-plane $y = 0$, $x < 0$ [13]. It has already been shown [19] that the unstable periodic orbits of the attractor can be generated with only one kneading sequence up to period 12 for parameters $c = 2$, $d = 4$ and $a = 0.408$ (corresponding to $L \geqslant 12$). We find similar results ($L \geqslant 9$) for $c = 2$, $d = 4$ and $0.125 < a < 0.415$. Table 2 shows the periodic windows up to period 9 in descending a order along with their periods, words and locations on the parameter axis. They are exactly consistent with part of the U-sequence from word C up to 1001011C.

Table 2. The symbolic sequences for the periodic windows of the Rössler's equations.

No.	Period	Word	a range
0	0	C	0.125–0.335
	2	1C	0.336–0.375
	2×2	101C	0.376–0.3834
	$2^2 \times 2$	1011101C	0.3836–0.3852
2	6	10111C	0.390 668–0.390 97
3	8	1011111C	0.393 624–0.393 638
4	9	10111111C	0.395 446–0.395 449
5	7	101111C	0.396 638–0.396 676
6	9	10111101C	0.398 034–0.398 041
7	5	1011C	0.399 948–0.399 966
8	9	10110101C	0.402 190–0.402 194
9	7	101101C	0.403 530–0.403 564
01	9	10110111C	0.404 690–0.404 691
00	8	1011011C	0.406 054
12	3	10C	0.409 12–0.410 91
	2×3	10010C	0.410 92–0.411 75
13	8	1001011C	0.414 432

The forced Brusselator had been extensively studied with the symbolic dynamics of 1D maps [4]. An incomplete U-sequence up to period 6 along the axis $A = 0.46 - 0.2\omega$ had already been found by Hao *et al* [4] which is also listed in table 1. Only the periodic window 10001C was missing. Our investigation on the Poincaré map with symbolic dynamics shows that $L = 2$ for the parameter range $0.8056 < \omega < 0.8194$ so that the U-sequence up to period 6 might be incomplete. Recently, Liu [21] has confirmed that the missing period 10001C is pruned.

5. Conclusion and discussion

In this paper the universal bifurcation property and its limitations of the Hénon map in 2D parameter space (a, b) is discussed with symbolic dynamics. Two topological quantities δ and L are defined to characterize this topological universality. In the universal-L region, as in the 1D unimodal map, there is a perfect MSS sequence up to period $\leqslant L$ along any axis provided that the axis is never tangent to any isoperiodic lines, though the phase portraits of the Hénon map exhibit very complicated 2D behaviour. Extending this idea to many other two- or higher-dimensional systems ensures that the symbolic dynamics of 1D mappings is an effective technique for investigating the universality in these two- or higher-dimensional systems and then the parameter for definite periodic motion may be predicted [22]. We have presented two examples of ordinary differential equations (ODEs), the Rössler equations [13] and the forced Brusselator [4], to demonstrate the validity and robustness of our approach.

It should be noted that only the short period is considered although the theory presented in this paper is also valid for higher period. In fact, in real experiments (or numerical studies of ODEs or PDEs), only short periodic orbits can be obtained. Our investigation shows that it is not surprising that universal sequences with short period are found in many experiments. Moreover, our result shows that the fact that universal sequences with short period can be found in many nonlinear dissipative systems is a universal phenomenon. This observation ensures that the parameters of many periodic motions for many dynamical systems (such as some fluid systems, e.g. [22]) can be well predicted.

In this paper we have also shown that even on an axis outside the universal region, the Hénon map can exhibit approximately 1D behaviour. This observation interprets the numerical results that in some nonlinear dynamical systems only incomplete U-sequences has been found [4]. Anyway, our defined universal-M region gives a background for interpreting the experimental and numerical observations that complete or imcomplete U-sequences with short period can be found in many dissipative systems and to understanding the limitations that 1D symbolic dynamics can be used to study two- or high-dimensional dissipative systems.

Acknowledgments

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References

[1] Metropolis N, Stein M and Stein P 1973 *J. Combinat. Theor.* A **16** 25
[2] Libchaber A, Fauve S and Laroche C 1983 *Physica* **7D** 73
[3] Coffman K, McCormick W D and Swinney H L 1986 *Phys. Rev. Lett.* **56** 999
Simoyi R H, Wolf A and Swinney H L 1982 *Phys. Rev. Lett.* 49 245
[4] Hao B L 1986 *Physica* **104A** 85
[5] Tomita K and Tsuda I 1980 *Prog. Theor. Phys. Suppl.* **69** 185
[6] Moore D R and Weiss N O 1990 *Phil. Trans. R. Soc.* A **332** 121
Proctor M R E and Weiss N O 1990 *Nonlinearity* 3 619
[7] Gunaratne G H, Jensen M H and Procaccia I 1988 *Nonlinearity* **1** 157
[8] Gunaratne G H, Linsay P S and Vinson M J 1989 *Phys. Rev. Lett.* **63** 0
[9] Fang H P and Hao B L 1995 *Symbolic Dynamics of the Lorenz Equations. Chaos, Soliton & Fractal* accepted
[10] Hénon M 1976 *Commun. Math. Phys.* **50** 65
[11] Mira C 1983 *Dynamical Systems and Chaos (Lecture Notes in Physics 179)* ed L Garrido (Berlin: Springer) and references therein
[12] Gallas J A C 1993 *Phys. Rev. Lett.* **70** 2714
[13] Rössler's O E 1976 *Phys. Lett.* **57A** 397
[14] Hao Bai-lin *Elementary Symbolic Dynamics and Chaos in Dissipative Systems* (Singapore: World Scientific)
[15] Milnor J and Thurston W 1988 *Lecture Notes in Math.* **1342** 463
[16] Cvitanović P, Gunaratne G H and Procaccia I 1988 *Phys. Rev.* A **38** 1503
[17] Grassberger P and Kantz H 1985 *Phys. Lett.* **113A** 235; 1985 *Physica* D **18** 75
[18] D'Alessandro G, Grassberger P, Isola S and Politi A 1990 *J. Phys. A: Math. Gen.* **23** 5295
[19] Fang H P 1994 *Phys. Rev.* E **49** 5025
[20] Flepp L, Holzner R, Brun E, Finardi M and Badii R 1991 *Phys. Rev. Lett.* **67** 2244
Finardi M, Flepp L, Parisi L, Holzner R, Badii R and Brun E 1992 *Phys. Rev. Lett.* **68** 2989
[21] Liu J X ongoing work, unpublished
[22] Fang H P and Liu Z H 1994 *Phys. Rev.* E **50** 2790